Prospects for Elasticity Reconstruction in the Heart

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***Abstract*—The elastic moduli in anisotropic media can be estimated using either direct mechanical or sound speed measurements. Here we compare moduli in the passive heart estimated with different methods and demonstrate that high-frequency (i.e., ultrasonic) sound speed measurements are inconsistent with static deformations and low-frequency shear wave results. Both tissue fixation and the high-operating frequency of ultrasonic measurements contribute to these discrepancies. Moreover, the precision of ultrasonic sound speed measurements required to estimate elastic moduli describing static deformations of a nearly incompressible anisotropic medium such as the heart appears to be beyond the scope of current methods. We conclude that an incompressible anisotropic elastic model is appropriate for elasticity reconstruction in the heart, in which three independent constants characterize small strain behavior, but four are needed for a fully nonlinear description of finite deformations.**

I. Introduction

To date, there have been limited studies of ultrasonic elasticity imaging in anisotropic media. With the advent of ultrasonic strain rate imaging of the heart, the problem of elastic modulus reconstruction in anisotropic media needs to be explored in more detail. Previous studies from two research communities have approached anisotropy quite differently.

In a classic set of papers, the anisotropic elastic properties of the passive (i.e., unperfused, nonbeating) heart have been analyzed using static force-deformation experiments and finite-element modeling [1]–[3]. These studies showed that the unperfused heart could be modeled as an incompressible, anisotropic medium with three independent elastic moduli describing the small strain (i.e., linear) behavior.

An alternate approach models the passive heart as a compressible, transversely orthotropic medium and uses ultrasonic wave speed measurements to estimate the five independent elastic moduli [4]–[7]. The results of these studies differ by several orders of magnitude from those presented in [1]–[3]. In this paper, we explore the elastic properties of a transversely orthotropic medium in the limit of incompressibility to reconcile the large differences in elastic moduli computed from ultrasonic and direct mechanical measurements. Our ultimate goal is to develop an elastic model appropriate for elastic modulus reconstruction in the heart using multidimensional ultrasonic strain and strain rate images. We start with a review of the elastic properties of both isotropic and transversely orthotropic media.

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II. Isotropic Medium

For isotropic materials, Hook's law takes the following simple form relating the stress tensor, σ, to the symmetric strain tensor, ε:

$$\sigma_{ij} = \lambda\theta\delta_{ij} + 2\mu\varepsilon_{ij},\ i,j,= 1,2,3, \tag{1}$$

where λ and μ are Lame coefficients representing the two independent elastic moduli fully describing the medium's elastic properties. Subscripts for both stress and strain refer to Cartesian coordinates. And, θ is the trace of the strain matrix, representing the divergence of the displacement vector.

To simplify later calculations for anisotropic materials, define a reduced notation for stress and strain tensors:

$$\begin{pmatrix} \sigma_1 = \sigma_{11} \\ \sigma_2 = \sigma_{22} \\ \sigma_3 = \sigma_{33} \\ \sigma_4 = \sigma_{13} = \sigma_{31} \\ \sigma_5 = \sigma_{23} = \sigma_{32} \\ \sigma_6 = \sigma_{12} = \sigma_{21} \end{pmatrix} \quad \begin{pmatrix} \varepsilon_1 = \varepsilon_{11} \\ \varepsilon_2 = \varepsilon_{22} \\ \varepsilon_3 = \varepsilon_{33} \\ \varepsilon_4 = \varepsilon_{13} = \varepsilon_{31} \\ \varepsilon_5 = \varepsilon_{23} = \varepsilon_{32} \\ \varepsilon_6 = \varepsilon_{12} = \varepsilon_{21} \end{pmatrix}. \tag{2}$$

With this notation, Hook's law takes the following matrix form:

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{pmatrix} = \begin{pmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\mu \end{pmatrix} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{pmatrix}. \tag{3}$$

These relations can be summarized as $\sigma = \mathbf{C}\varepsilon$, where C is the elastic modulus matrix. The full set of equations reduces to two independent 3×3 matrix equations for longitudinal ($\mathbf{C_l}$) and shear ($\mathbf{C_s}$) components:

$$\begin{pmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\end{pmatrix} = \begin{pmatrix}\lambda+2\mu & \lambda & \lambda\\ \lambda & \lambda+2\mu & \lambda\\ \lambda & \lambda & \lambda+2\mu\end{pmatrix} \begin{pmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\end{pmatrix}, \quad (4)$$

$$\begin{pmatrix}\sigma_4\\ \sigma_5\\ \sigma_6\end{pmatrix} = \begin{pmatrix}2\mu & 0 & 0\\ 0 & 2\mu & 0\\ 0 & 0 & 2\mu\end{pmatrix} \begin{pmatrix}\varepsilon_4\\ \varepsilon_5\\ \varepsilon_6\end{pmatrix}. \quad (5)$$

The incompressibility condition is a constraint on the strain, specifically the trace of the strain matrix. Consequently, we also must express the strain in terms of the stress to define the proper limits on elastic moduli for incompressible media. To solve for the strain, Hook's law must be inverted (i.e., $\varepsilon = \mathbf{C^{-1}}\sigma$), where $\mathbf{C^{-1}}$ is the inverse of the elastic modulus matrix (i.e., compliance). For shear terms, the inversion is trivial. For longitudinal terms, the inverse of the 3×3 matrix $\mathbf{C_l}$ is:

$$\mathbf{C_l^{-1}} = \frac{1}{\Delta}\begin{pmatrix}2(\lambda+\mu) & -\lambda & -\lambda\\ -\lambda & 2(\lambda+\mu) & -\lambda\\ -\lambda & -\lambda & 2(\lambda+\mu)\end{pmatrix}, \quad (6)$$

where Δ is the determinant of $\mathbf{C_l}$ divided by the common factor 2μ, and is given by:

$$\Delta = 2\mu(3\lambda+2\mu). \quad (7)$$

This expression can be further reduced into a more familiar form by noting that the Young's modulus E can be defined in terms of the Lame constants as:

$$E = \frac{\mu(3\lambda+2\mu)}{(\lambda+\mu)}, \quad (8)$$

and Poisson's ratio ν can be defined similarly in terms of the Lame constants as:

$$\nu = \frac{\lambda}{2(\lambda+\mu)}. \quad (9)$$

Given these definitions, $\mathbf{C_l^{-1}}$ reduces to:

$$\mathbf{C_l^{-1}} = \frac{1}{E}\begin{pmatrix}1 & -\nu & -\nu\\ -\nu & 1 & -\nu\\ -\nu & -\nu & 1\end{pmatrix}. \quad (10)$$

Consequently, for an isotropic compressible medium, the longitudinal elastic properties can be fully characterized by two sets of complementary parameters: $[\lambda, \mu]$ or $[E, \nu]$ [8].

III. Transversely Orthotropic Medium

The elastic modulus matrix for a medium with transversely orthotropic symmetry—as would be found in a muscle fiber, for instance, in which the fiber axis is defined as the z (i.e., $i = 3$) axis—can be written in terms of five independent elastic moduli:

$$\mathbf{C} = \begin{pmatrix}C_{11} & C_{12} & C_{13} & 0 & 0 & 0\\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0\\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0\\ 0 & 0 & 0 & 2C_{44} & 0 & 0\\ 0 & 0 & 0 & 0 & 2C_{44} & 0\\ 0 & 0 & 0 & 0 & 0 & (C_{11}-C_{12})\end{pmatrix}. \quad (11)$$

Note that the $i = 1, 2$ dimensions are isotropic and, consequently, C_{11} can be recognized as equivalent to $(\lambda + 2\mu)$, C_{12} equivalent to λ, and $(C_{11} - C_{12})$ equivalent to 2μ. Consequently, the five independent elastic moduli for this symmetry group are $[\lambda, \mu, C_{13}, C_{33}, C_{44}]$, the longitudinal elastic matrix $\mathbf{C_l}$ can be written as:

$$\mathbf{C_l} = \begin{pmatrix}\lambda+2\mu & \lambda & C_{13}\\ \lambda & \lambda+2\mu & C_{13}\\ C_{13} & C_{13} & C_{33}\end{pmatrix}, \quad (12)$$

and the shear elastic matrix $\mathbf{C_s}$ can be written as:

$$\mathbf{C_s} = \begin{pmatrix}2\mu_{13} & 0 & 0\\ 0 & 2\mu_{13} & 0\\ 0 & 0 & 2\mu\end{pmatrix}, \quad (13)$$

where $\mu_{13} = C_{44}$ is an alternate notation often used for the transverse-axial shear modulus. This means the longitudinal elastic properties can be characterized fully by the four independent moduli $[\lambda, \mu, C_{13}, C_{33}]$.

To solve for the strain, Hook's law must be inverted (i.e., $\varepsilon = \mathbf{C^{-1}}\sigma$), where $\mathbf{C^{-1}}$ is the inverse of the elastic modulus matrix. For shear terms, the inversion is again trivial. For longitudinal terms, the inverse of the 3×3 matrix $\mathbf{C_l}$ is:

$$\mathbf{C_l^{-1}} = \frac{1}{\Delta}\begin{pmatrix}\alpha_{11} & -\alpha_{12} & -\alpha_{13}\\ -\alpha_{12} & \alpha_{11} & -\alpha_{13}\\ -\alpha_{13} & -\alpha_{13} & \alpha_{33}\end{pmatrix}, \quad (14)$$

where the matrix elements are defined as:

$$\begin{aligned}\Delta &= 4\mu\left(C_{33}(\lambda+\mu) - C_{13}^2\right),\\ \alpha_{11} &= C_{33}(\lambda+2\mu) - C_{13}^2,\\ \alpha_{12} &= C_{33}\lambda - C_{13}^2,\\ \alpha_{13} &= C_{13}(\lambda+2\mu-\lambda) = 2\mu C_{13},\\ \alpha_{33} &= (\lambda+2\mu)^2 - \lambda^2 = 4\mu(\lambda+\mu).\end{aligned} \quad (15)$$

Similar to the isotropic case, the matrix elements can be related to equivalent Young's moduli and Poisson's ratios such that $\mathbf{C_l^{-1}}$ becomes:

$$\mathbf{C_l^{-1}} = \begin{pmatrix}\frac{1}{E_1} & \frac{-\nu_{12}}{E_1} & \frac{-\nu_{13}}{E_3}\\ \frac{-\nu_{12}}{E_1} & \frac{1}{E_1} & \frac{-\nu_{13}}{E_3}\\ \frac{-\nu_{13}}{E_3} & \frac{-\nu_{13}}{E_3} & \frac{1}{E_3}\end{pmatrix}, \quad (16)$$

where

$$\begin{aligned} E_1 &= \Delta/\alpha_{11}, \\ E_3 &= \Delta/\alpha_{33}, \\ \nu_{12} &= \alpha_{12}/\alpha_{11}, \\ \nu_{13} &= \alpha_{13}/\alpha_{33}. \end{aligned} \tag{17}$$

Consequently, for a transversely orthotropic compressible medium, the longitudinal elastic properties also can be characterized fully by the four independent parameters $[E_1, E_3, \nu_{12}, \nu_{13}]$.

In the next section we explore how these equations can be reduced for the case of incompressible media. Note that we will not consider the degenerate case where $\Delta = 0$ for a transversely orthotropic medium. This is a very special case that will not happen for normal elastic materials considered in biomechanics. In the limit of a transversely orthotropic medium approaching an isotropic medium (i.e., $C_{13} \to \lambda$, $C_{33} \to \lambda + 2\mu$), Δ is nonzero.

IV. Incompressible Media

Under the restriction of a small deformation (i.e., linear elasticity) the incompressibility condition is:

$$\theta = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = 0. \tag{18}$$

This condition does not depend on the details of the applied stress and simply is the differential form of volume conservation (i.e., $\Delta V = 0$). For an isotropic material, applying this condition to the strain using $\boldsymbol{\varepsilon} = \mathbf{C^{-1}}\sigma$ and (10) for the inverse elastic modulus matrix yields the following equation for θ:

$$\begin{aligned} \theta &= \frac{1}{E}\left[\sigma_{11}(1-2\nu) + \sigma_{22}(1-2\nu) + \sigma_{33}(1-2\nu)\right] = 0, \\ \theta &= \frac{3(1-2\nu)}{E}\sigma = 0, \end{aligned} \tag{19}$$

where $\sigma = [\sigma_{11}+\sigma_{22}+\sigma_{33}]/3$ is the mean internal pressure. This means that $\nu = 1/2$ to ensure that $\theta = 0$ for an arbitrary load.

Imposing (18), the limits of the elastic moduli can be explored using the relations between the Lame constants and the Young's modulus and Poisson's ratio:

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \ \mu = \frac{E}{2(1+\nu)}. \tag{20}$$

For an incompressible medium, $\nu \to 1/2$, $\mu \to E/3$, $\lambda \to \infty$, and, according to (19) and (20), the product $\lambda\theta$ equals σ, usually defined as the scalar pressure P. Therefore, (1) for an incompressible medium reduces to:

$$\sigma_{ij} = P\delta_{ij} + 2\mu\varepsilon_{ij}, \ i, j, = 1, 2, 3, \tag{21}$$

where $P = \lim(\lambda\theta)$ when $\lambda \to \infty$ and $\theta \to 0$. Note that μ must be finite to ensure a finite shear elastic matrix, which means that the Young's modulus also must be finite. Thus, according to (20) and (21), only one material parameter (μ or E, where $E = 3\mu$) is needed to describe the behavior of an isotropic incompressible medium, but in this case we have the additional scalar unknown P.

Let us repeat the analysis for an incompressible transversely orthotropic medium. If $\Delta \neq 0$, from (14) and (15), the incompressibility condition can be expressed as:

$$\theta = \frac{\alpha_{11} - \alpha_{12} - \alpha_{13}}{\Delta}(\sigma_{11} + \sigma_{22}) + \frac{\alpha_{33} - 2\alpha_{13}}{\Delta}\sigma_{33} = 0. \tag{22}$$

Define the two new variables:

$$\begin{aligned} Q_1 &= (C_{13} - \lambda), \\ Q_2 &= [C_{33} - (\lambda + 2\mu)]. \end{aligned} \tag{23}$$

Note that for an isotropic material $Q_1 = Q_2 = 0$. Using these new variables and the definitions of the inverse elastic modulus matrix components, the terms in the incompressibility condition of (22) take the form:

$$\begin{aligned} &\alpha_{11} - \alpha_{12} - \alpha_{13} = 2\mu(2\mu + Q_2 - Q_1), \\ &\alpha_{33} - 2\alpha_{13} = 4\mu(\mu - Q_1), \\ &\Delta = 4\mu[\lambda(3\mu + Q_2 - 2Q_1) + (2\mu^2 + \mu Q_2 - Q_1^2)]. \end{aligned} \tag{24}$$

Therefore, we can conclude that the incompressibility condition is satisfied for any σ_{11}, σ_{22}, and σ_{33} if $\lambda \to \infty$ while μ, Q_1, and Q_2 are finite. In this case, the longitudinal terms reduce to:

$$\begin{aligned} \sigma_{11} &= P + 2\mu\varepsilon_{11} + Q_1\varepsilon_{33}, \\ \sigma_{22} &= P + 2\mu\varepsilon_{22} + Q_1\varepsilon_{33}, \\ \sigma_{33} &= P + 2\mu\varepsilon_{33} + Q_1(\varepsilon_{11} + \varepsilon_{22}) + Q_2\varepsilon_{33}, \end{aligned} \tag{25}$$

where $P = \lim(\lambda\theta)$ when $\theta \to 0$. These formulas are the stress-strain relations for an incompressible transversely orthotropic medium, in which the elastic behavior is fully characterized by the four material constants [μ, μ_{13}, Q_1, Q_2] and the scalar pressure P.

The longitudinal elastic properties of a compressible, transversely orthotropic medium also can be fully described by the Young's moduli and the Poisson's ratios, in which there is a one-to-one correspondence between these parameters and the elastic moduli of Hook's law:

$$\begin{aligned} \lambda &= E_1(E_3\nu_{12} + E_1\nu_{13}^2)/\delta, \\ \mu &= E_1[E_3(1-\nu_{12}) - 2E_1\nu_{13}^2]/2\delta, \\ C_{13} &= E_1E_3\nu_{13}(1+\nu_{12})/\delta, \\ C_{33} &= E_3^2(1-\nu_{12}^2)/\delta, \\ \delta &= (1+\nu_{12})[E_3(1-\nu_{12}) - 2E_1\nu_{13}^2]. \end{aligned} \tag{26}$$

Similarly, the two elastic moduli defined above to simplify notation for an incompressible medium are:

$$\begin{aligned} Q_1 &= (C_{13} - \lambda) = E_1[E_3(\nu_{13} + \nu_{12}\nu_{13} - \nu_{12}) - E_1\nu_{13}^2]/\delta, \\ Q_2 &= [C_{33} - (\lambda + 2\mu)] = [E_3^2(1-\nu_{12}^2) - E_1E_3 + E_1^2\nu_{13}^2]/\delta. \end{aligned} \tag{27}$$

Using these alternate definitions, the incompressibility condition of (22) becomes:

$$\theta = [(1-\nu_{12})/E_1-\nu_{13}/E_3](\sigma_{11}+\sigma_{22})+(1-2\nu_{13})\sigma_{33}/E_3=0. \tag{28}$$

For this equation to be satisfied for an arbitrary load (i.e., arbitrary σ_{11}, σ_{22}, or σ_{33}), the following two conditions must be satisfied simultaneously:

$$\nu_{13}=\frac{1}{2},\ \upsilon_{12}=1-\frac{E_1}{2E_3}. \tag{29}$$

Note if the medium is isotropic, then $E_1 = E_3 = E$, and these two conditions lead to $\nu_{13}=\nu_{12}=\nu=\frac{1}{2}$. For normal materials found in biomechanics, the requirement $\nu > 0$ seems to be valid. This requirement is automatically satisfied for the limit value of $\nu_{13}=\frac{1}{2}$, but for the limit value of $\nu_{12}=1-\frac{E_1}{2E_3}$ only if $E_1 \le 2E_3$.

Using the Young's moduli and Poisson's ratios to describe a transversely orthotropic material, it appears that the requirement of medium incompressibility reduces the number of material parameters needed to describe longitudinal components to two: $[E_1, E_3]$. Unfortunately, (29) cannot be directly used in the stress-strain relations. Also, note that δ is zero for this case, signaling that the conversion between the two sets of elastic moduli presented in (23) is not defined. Therefore, the limit behavior of coefficients in (26) and (27) when $(\nu_{12}, \nu_{13}) \to \left(1-\frac{E_1}{2E_3}, \frac{1}{2}\right)$ needs some additional consideration.

To consider the limit case of incompressibility, use the local cylindrical system of coordinates in the plane (ν_{12}, ν_{13}) with the center $\left(1-\frac{E_1}{2E_3}, \frac{1}{2}\right)$. That is, assume that:

$$\nu_{12}=\left(1-\frac{E_1}{2E_3}\right)-rC,\ \nu_{13}=\frac{1}{2}-rS, \tag{30}$$

where $C=\cos(\alpha)$, $S=\sin(\alpha)$. If a medium is nearly incompressible, the value of r is small, and for incompressible medium the limit $r \to 0$ must be evaluated.

Using (26), (27), and (30), for small r we obtain:

$$\begin{aligned}
\delta &= r[(4E_3-E_1)-2rCE_3][(E_3C+2E_1S)-2E_1rS^2]/2E_3,\\
\lambda &= E_1(E_3\nu_{12}+E_1\nu_{13}^2)/\delta\\
&= \frac{E_3E_1[(4E_3-E_1)-4r(E_3C+E_1S)+4r^2E_1S^2]}{2r[(4E_3-E_1)-2rCE_3][(E_3C+2E_1S)-2E_1rS^2]},\\
\mu &= E_1[E_3(1-\nu_{12})-2E_1\nu_{13}^2]/2\delta\\
&= \frac{E_1E_3}{[(4E_3-E_1)-2rCE_3]},\\
Q_1 &= E_1[E_3(\nu_{13}+\nu_{12}\nu_{13}-\nu_{12})-E_1\nu_{13}^2]/\delta\\
&= \frac{E_3E_1\{[E_3(C-4S)+3E_1S]+2rS(E_3C-E_1S)\}}{[(4E_3-E_1)-2rCE_3][(E_3C+2E_1S)-2E_1rS^2]},\\
Q_2 &= [E_3^2(1-\nu_{12}^2)-E_1E_3+E_1^2\nu_{13}^2]/\delta\\
&= \frac{2E_3[(2E_3^2C-E_1E_3C-E_1^2S)+r(E_1^2S^2-E_3^2C^2)]}{[(4E_3-E_1)-2rCE_3][(E_3C+2E_1S)-2E_1rS^2]}.
\end{aligned} \tag{31}$$

These expressions have different behavior when $r \to 0$ for different values of $(4E_3-E_1)$. If $(4E_3-E_1) \neq 0$ from (31) the elastic moduli become:

$$\begin{aligned}
\lambda &\approx \frac{E_3E_1}{2r(E_3C+2E_1S)} \to \infty,\\
\mu &\to \frac{E_1E_3}{(4E_3-E_1)},\\
Q_1 &\to \frac{E_3E_1[E_3(1-4t)+3E_1t]}{(4E_3-E_1)(E_3+2E_1t)},\\
Q_2 &\to \frac{2E_3(2E_3^2-E_1E_3-E_1^2t)}{(4E_3-E_1)(E_3+2E_1t)},
\end{aligned} \tag{32}$$

where $t=S/C=\tan(\alpha)$.

In contrast, if $(4E_3-E_1)=0$, from (31) we have:

$$\begin{aligned}
\lambda &= \frac{4E_3[(C+4S)-4rS^2]}{rC[(C+8S)-8rS^2]} \to \infty,\\
\mu &= -\frac{2E_3}{rC} \to \infty,\\
Q_1 &= -\frac{2E_3[(C+8S)+2rS(C-4S)]}{rC[(C+8S)-8rS^2]} \to \infty,\\
Q_2 &= \frac{E_3[2(C+8S)-r(16S^2-C^2)]}{rC[(C+8S)-8rS^2]} \to \infty.
\end{aligned} \tag{33}$$

Note that if $(4E_3-E_1)=0$ for an incompressible medium, from (30) $\nu_{12}=-1$ and $\mu=\infty$. From (33) we conclude that, if $(4E_3-E_1)=0$, the set of parameters $(E_1, E_3, \nu_{12}, \nu_{13})$ cannot be used to derive the stress-strain relation for an incompressible transversely orthotropic medium. If $(4E_3-E_1)\neq 0$, the last terms in (32) have finite values when $r \to 0$. These values, however, depend on the angle α in the (ν_{12}, ν_{13}) plane, which was used to reach the limit point $(\nu_{12}, \nu_{13}) = \left(1-\frac{E_1}{2E_3}, \frac{1}{2}\right)$. Therefore, in general, the limit case of an incompressible transversely orthotropic medium cannot be described using (29) and material parameters $[E_1, E_3]$. This fact can lead to the mistaken conclusion that a transversely orthotropic medium cannot be incompressible. The correct conclusion seems to be that the stress-strain relations in an incompressible transversely orthotropic medium generally cannot be described using only a total of three independent material parameters. The angle α [or parameter t used in (32)] is the fourth material parameter needed to describe the complete stress-strain relation. As shown below, however, in many physically realizable cases, an incompressible, transversely orthotropic material can be well characterized using only three elastic moduli.

V. Approximating Incompressible Transversely Orthotropic Media

Using an isotropic material as the limit case, we can approximate the proper angle to derive a one-to-one correspondence between the complementary descriptions of an incompressible, transversely orthotropic medium. The relations in (31) must contain the relations for an incompressible, isotropic medium as a limit case. For this case, (31) reduces to:

$$\begin{aligned} &\lambda \approx E/2r(C+2S),\\ &\mu \approx E/3,\\ &Q_1 \approx E(C-S)/3(C+2S),\\ &Q_2 \approx 2E(C-S)/3(C+2S). \end{aligned} \tag{34}$$

From (34) we can conclude that, for an isotropic medium as $r \to 0$, (34) reduces to $\lambda \to \infty$, $\mu = E/3$, $Q_1 = 0$, and $Q_2 = 0$ if $C = S$ (i.e., $t = 1$). Now, keep this condition for an anisotropic medium. With this limitation we can rewrite (32) in the form:

$$\begin{aligned} &\lambda \to \infty,\\ &\mu \to \frac{E_1E_3}{(4E_3-E_1)},\\ &Q_1 \to \frac{3E_3E_1(E_1-E_3)}{(4E_3-E_1)(E_3+2E_1)},\\ &Q_2 \to -\frac{2E_3(E_1-E_3)(E_1+2E_3)}{(4E_3-E_1)(E_3+2E_1)}. \end{aligned} \tag{35}$$

Therefore, if $(4E_3 - E_1) \neq 0$, the stress-strain relations for an incompressible, transversely orthotropic medium become:

$$\begin{aligned} \sigma_{11} &= P + \frac{2E_1E_3}{(4E_3-E_1)}\varepsilon_{11} + \frac{3E_1E_3(E_1-E_3)}{(4E_3-E_1)(E_3+2E_1)}\varepsilon_{33},\\ \sigma_{22} &= P + \frac{2E_1E_3}{(4E_3-E_1)}\varepsilon_{22} + \frac{3E_1E_3(E_1-E_3)}{(4E_3-E_1)(E_3+2E_1)}\varepsilon_{33},\\ \sigma_{33} &= P + \frac{2E_1E_3}{(4E_3-E_1)}\varepsilon_{33}\\ &\quad + \frac{3E_1E_3(E_1-E_3)}{(4E_3-E_1)(E_3+2E_1)}(\varepsilon_{11}+\varepsilon_{22})\\ &\quad - \frac{2E_3(E_1-E_3)(E_1+2E_3)}{(4E_3-E_1)(E_3+2E_1)}\varepsilon_{33},\\ \sigma_{13} &= 2\mu_{13}\varepsilon_{13},\\ \sigma_{23} &= 2\mu_{13}\varepsilon_{23},\\ \sigma_{12} &= \frac{2E_1E_3}{(4E_3-E_1)}\varepsilon_{12}. \end{aligned} \tag{36}$$

These formulas can be considered as stress-strain relations for transversely orthotropic, incompressible medium, written using the second set of material parameters. Clearly, only three material parameters (E_1, E_3, μ_{13}) are needed, where again we have the additional scalar unknown P. Expressions (36) contain (21) as a limit case for isotropic medium when $E_1 = E_2 = E$ and $\mu_{13} = \mu$. Note also that $(4E_3 - E_1) \neq 0$ if $\nu_{12} > 0$, i.e., if $E_1 \leq 2E_3$. At the same time, we must remember that (36) was obtained using the strong assumption $C = S$. Unfortunately for highly anisotropic materials, this assumption is not obvious and must be considered as an additional strong limitation. To describe the elastic behavior of an incompressible, transversely orthotropic medium using (36), it is necessary that this limitation be justified using force-deformation measurements on the material of interest.

TABLE I
ELASTIC MODULI OF PASSIVE DOG MYOCARDIUM ESTIMATED FOR DIRECT MECHANICAL MEASUREMENTS ASSUMING AN INCOMPRESSIBLE MEDIUM (ADAPTED FROM [2], [3]).

Modulus	Value (kPa)
Q_1	0
Q_2	13.1
μ_{13}	3.2
μ	1.4

VI. Discussion

Guccione *et al.* [2], [3] computed three independent elastic moduli for the passive heart based on optimal fits to force-deformation measurements in intact dog myocardium. Their model was incompressible and should have produced four independent elastic moduli. Nonlinear, least-squares fits, however, were optimized by setting one of the coefficients to zero (Q_1 in our notation). Taking the small strain limit of their nonlinear, constitutive relations, and converting to our notation, the four independent moduli estimated from their results are summarized in Table I.

These values represent the zero strain limits of the nonlinear, constitutive relations and should be considered the lower bound on the magnitude of the moduli in the linear elastic regime.

Recent work in transient elastography has investigated shear wave speeds at very low propagation frequencies (100 Hz) to estimate anisotropy in the shear elastic moduli of striated muscle [9]. Measurements in the human biceps exhibited anisotropy (a factor of 4 in wave speed, representing a factor of 16 in the shear elastic moduli) greater than that reported in myocardium, but the magnitude of the shear modulus (μ = 9 kPa) is consistent with the results reported in Table I, given that these measurements were obtained with the muscle loaded (i.e., not in the zero strain limit for this highly nonlinear material). Overall, both low frequency shear wave and static deformation measurements strongly suggest that all elastic moduli describing static deformation of the passive heart in the linear regime should be on the order of 10 to 20 kPa.

In materials characterization, elastic moduli in an anisotropic material are often estimated from ultrasonic sound speed measurements. These moduli then are used to describe static deformations of the material. Similar methods have been applied to the passive heart to produce estimates of anisotropic cardiac elastic moduli [4]–[7]. Measurements of ultrasonic sound speeds at high frequencies (5 MHz) in formalin-fixed human myocardium produce values for the five elastic moduli of a transversely orthotropic, compressible model, as presented in Table II.

TABLE II
Elastic Moduli of Passive Dog Myocardium Estimated from Ultrasonic Velocity Measurements Assuming a Compressible Medium (adapted from [4]–[7]).

Modulus	Value (GPa)
C_{11}	2.462
C_{33}	2.527
C_{13}	2.445
C_{44}	9.0×10^{-3}
C_{66}	8.5×10^{-3}

Note that the shear elastic moduli are more than three orders of magnitude greater than those estimated using either low-frequency shear wave or static deformation measurements. Formalin fixation clearly contributes to the difference; previous studies noted that fixation can increase the static shear modulus by at least an order of magnitude [10]. Also, measurements of shear properties at high frequencies are very hard to extrapolate to low frequencies given the highly dispersive nature of shear wave propagation in soft tissue [11].

Using these values, the equivalent elastic moduli of an incompressible model can be computed. An incompressible model should yield reasonable results as ν_{13} is 0.4975 for these parameters, closely approximating the incompressible value of 0.5. The two moduli Q_1 and Q_2 describing anisotropies in the longitudinal elastic matrix are:

$$\begin{aligned} Q_1 &= (C_{13} - \lambda) = (C_{13} - C_{11} + C_{66}) \\ &= (2.445 - 2.462 + 0.0085)\, GPa \\ &= -0.0085\, GPA = -8.5\, MPa. \\ Q_2 &= (C_{33} - C_{11}) = (2.527 - 2.462)\, GPa \\ &= 0.065\, GPa = 65.00\, MPa. \end{aligned} \tag{37}$$

Note that Q_1 is negative. If C_{11} were 2.4535 GPa, representing a 0.35% change in the modulus (0.17% change in the sound speed), then Q_1 would be zero, matching the static results. As noted in [4], ultrasonic wave speed measurements on well controlled tissue samples are precise only to about 0.20%. Consequently, it is virtually impossible to yield a reasonable estimate of Q_1 using these parameters. Similarly, because computation of Q_2 requires the difference in elastic moduli computed directly from wave speed measurements, the error in this parameter is nearly 40%. With this level of error, it is nearly impossible to judge the relative contributions of different elastic moduli let alone the absolute magnitudes. And, to estimate Q_2 from the longitudinal wave speed to a precision of 10 kPa, the speed must be measured to a precision of about two parts per million. This is beyond the scope of conventional ultrasonic measurements on soft tissue samples. Consequently, it is very difficult, and virtually impossible in most cases, to accurately compute elastic moduli from ultrasonic sound speed measurements that describe static soft tissue deformations.

Both static deformation experiments and ultrasonic sound speed measurements confirm that the passive heart can be modeled as an incompressible material. This assumption was justified originally over 20 years ago by Vossoughi *et al.* [12] and has helped improve the accuracy of multidimensional, ultrasonic speckle tracking algorithms [13]–[15]. More recent studies suggest that, even in the perfused heart, volume conservation at the regional level is a good approximation [16]. Assuming both incompressibility and modulus values derived from measurements during diastole, reconstructed strains differ from true values by at most 10% in peak systole. Given the large dynamic range in cardiac elastic moduli due to pathological processes, a 10% error from model imperfection seems like a very small price to pay to gain the signal-to-noise ratio benefits of incompressibility processing.

However, a complete description of cardiac elastic behavior must include anisotropic moduli in which four independent constants are required theoretically for an incompressible, transversely orthotropic medium, but three independent constants appear adequate to model small strain behavior. Note that the ratio of Q_2 to μ is nearly an order of magnitude; consequently, significant reconstruction errors will result if a single elastic modulus (i.e., E or μ) is assumed. The model of cardiac mechanics originally proposed by Guccione *et al.* [3] uses three parameters similar to E_1, E_3, and μ_{13} to describe small strain behavior, and four parameters to describe nonlinear behavior. The strain energy potential W for this model written in terms of the orthotropic coordinate axes is:

$$\begin{aligned} W &= \left(\frac{C}{2}\right)(e^Q - 1), \\ Q &= b_f(\varepsilon_{33}^2) + b_i(\varepsilon_{33}^2 + \varepsilon_{22}^2 + 2\varepsilon_{12}^2) + 2b_{fs}(\varepsilon_{13}^2 + \varepsilon_{23}^2). \end{aligned} \tag{38}$$

where C, b_f, b_i, and b_{fs} are the four independent material constants. Elasticity reconstruction algorithms based on ultrasonic strain and strain rate measurements in the heart must produce the spatial distribution of these parameters. Also, any elasticity imaging algorithm also must reconstruct the orientation of the fiber axis (i.e., orthotropic coordinate axis) at every position within the heart with respect to the absolute coordinate frame used for deformation analysis and modulus reconstruction.

VII. Summary

Elastic moduli in the passive heart estimated by different methods were compared. Moduli obtained from ultrasonic sound speed measurements are inconsistent with those obtained by static deformation and low-frequency shear wave methods. Both tissue fixation and the high-operating frequency of ultrasonic measurements contribute to these discrepancies. Moreover, the precision of ultrasonic sound speed measurements required to estimate elastic moduli describing static deformations of a nearly incompressible, anisotropic medium such as the heart appears to be beyond the scope of current methods. We conclude that an incompressible, anisotropic elastic model is appropriate for elasticity reconstruction in the heart, in which three independent constants characterize small strain behavior, but four are needed for a fully nonlinear description of finite deformations.

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